

4D singular oscillator and generalized MIC-Kepler system

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Abstract

It is shown that the generalized MIC-Kepler system and four-dimensional singular oscillator are dual to each other and the duality transformation is the generalized version of the Kustaanheimo-Stiefel transformation.

The Schrödinger equation for a generalized MIC-Kepler or charge-dyon system has the form [1]

$$\frac{1}{2\mu} \left(\hat{p}_i + \frac{e}{c} A_i^{(\pm)} \right)^2 \psi^{(\pm)} + \left[\frac{\hbar^2 s^2}{2\mu r^2} - \frac{e^2}{r} + \frac{\lambda_1}{r(r+z)} + \frac{\lambda_2}{r(r-z)} \right] \psi^{(\pm)} = E \psi^{(\pm)}, \quad (1)$$

where λ_1 and λ_2 are nonnegative constants. We recall that a dyon is a hypothetical particle introduced by Schwinger [2] and is a source of both an electric and magnetic field. The vector potentials

$$\mathbf{A}^{(\pm)} = \frac{1}{r(r \mp z)} (\pm y, \mp x, 0)$$

correspond to a Dirac monopole [3] with the magnetic charge $g = \hbar cs/e$ ($s = 0, \pm 1/2, \pm 1, \dots$) and with the axes $z > 0$ and $z < 0$ correspondingly. It is easily seen that the vector potentials $A_i^{(+)}$ and $A_i^{(-)}$ are connected by a gauge transformation

$$A_i^{(-)} = A_i^{(+)} + \frac{\partial f}{\partial x_i},$$

where $f = 2g \arctan(y/x)$ and the strength of the dyon magnetic field is

$$\mathbf{B} = \nabla \times \mathbf{A}^{(\pm)} = g \frac{\mathbf{r}}{r^3}.$$

It should be noted that the Schrödinger equation (1) for $\lambda_1 = \lambda_2 = 0$ and $s = 0$ reduces to the Schrödinger equation of the MIC-Kepler system [4, 5]. At $s = 0$ Eq. (1) is reduced to the Schrödinger equation for the generalized Kepler-Coulomb problem [6]. In case when $s = 0$ and $c_1 = c_2 \neq 0$, the equation (1) reduces to the Hartmann system that has been used for describing axially symmetric systems like ring-shaped molecules [7].

In [1, 8] it is shown that the variables in the Schrödinger equation (1) are separated in spherical, parabolic and prolate spheroidal coordinates. For completeness, here we present

the explicit forms of the spherical and parabolic bases of the generalized MIC-Kepler system found in [1].

In the spherical coordinates

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta \quad (2)$$

the wave function of the generalized MIC-Kepler system has the form

$$\psi \equiv \psi_{njm}^{(s)}(r, \theta, \varphi; \delta_1, \delta_2) = R_{nj}^{(s)}(r; \delta_1, \delta_2) Z_{jm}^{(s)}(\theta, \varphi; \delta_1, \delta_2). \quad (3)$$

The functions $Z_{jm}^{(s)}(\theta, \varphi; \delta_1, \delta_2)$ and $R_{nj}^{(s)}(r; \delta_1, \delta_2)$ are given by the formulae

$$Z_{jm}^{(s)}(\theta, \varphi; \delta_1, \delta_2) = N_{jm}(\delta_1, \delta_2) \left(\cos \frac{\theta}{2} \right)^{m_1} \left(\sin \frac{\theta}{2} \right)^{m_2} P_{j-m_+}^{(m_2, m_1)}(\cos \theta) e^{i(m+s)\varphi},$$

$$R_{nj}^{(s)}(r) = C_{nj}(\delta_1, \delta_2) (2\varepsilon r)^{j+\frac{\delta_1+\delta_2}{2}} e^{-\varepsilon r} F(-n+j+1; 2j+\delta_1+\delta_2+2; 2\varepsilon r),$$

where $P_n^{(\alpha, \beta)}(x)$ are the Jacobi polynomials, $F(a; c; x)$ is the confluent hypergeometric function, $N_{jm}(\delta_1, \delta_2)$ and $C_{nj}(\delta_1, \delta_2)$ are normalization constants

$$N_{jm}(\delta_1, \delta_2) = \sqrt{\frac{(2j+\delta_1+\delta_2+1)(j-m_+)!\Gamma(j+m_+\delta_1+\delta_2+1)}{4\pi\Gamma(j-m_-\delta_1+1)\Gamma(j+m_-\delta_2+1)}},$$

$$C_{nj}(\delta_1, \delta_2) = \frac{2\varepsilon^2}{\Gamma(2j+\delta_1+\delta_2+2)} \sqrt{\frac{\Gamma(n+j+\delta_1+\delta_2+1)}{(n-j-1)!}}.$$

We denote the following expression by ε :

$$\varepsilon = \sqrt{-\frac{2\mu E}{\hbar^2}} = \frac{1}{a \left(n + \frac{\delta_1+\delta_2}{2} \right)},$$

where $a = \hbar^2/\mu e^2$ is the Bohr radius. The energy spectrum has the form

$$E \equiv E_n^{(s)} = -\frac{\mu e^4}{2\hbar^2 \left(n + \frac{\delta_1+\delta_2}{2} \right)^2} \quad (4)$$

and the quantum numbers m and j run through the values: $m = -j, -j+1, \dots, j-1, j$ and $j = m_+, m_++1, \dots, n-1$. We also make the following notation:

$$m_{\pm} = \frac{|m+s| \pm |m-s|}{2}, \quad m_{1,2} = |m \pm s| + \delta_{1,2} = \sqrt{(m \pm s)^2 + \frac{4\mu\lambda_{1,2}}{\hbar^2}}.$$

The wave functions (3) are the eigenfunctions of commuting operators \hat{M} and \hat{J}_z and

$$\hat{M} \psi_{njm}^{(s)}(r, \theta, \varphi; \delta_1, \delta_2) = \left(j + \frac{\delta_1+\delta_2}{2} \right) \left(j + \frac{\delta_1+\delta_2}{2} + 1 \right) \psi_{njm}^{(s)}(r, \theta, \varphi; \delta_1, \delta_2),$$

where

$$\hat{M} = \hat{J}^2 + \frac{2c_1}{1 + \cos \theta} + \frac{2c_2}{1 - \cos \theta}.$$

Here \hat{J}^2 is the square of the angular momentum [4]

$$\hat{\mathbf{J}} = \frac{1}{\hbar} \left[\mathbf{r} \times \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \right] - s \frac{\mathbf{r}}{r},$$

$\hat{J}_z = -(s + i\partial/\partial\varphi)$ its z -component and $\hat{J}_z \psi^{(s)} = m\psi^{(s)}$.

In the parabolic coordinates

$$x = \sqrt{\xi\eta} \cos \varphi, \quad y = \sqrt{\xi\eta} \sin \varphi, \quad z = \frac{1}{2}(\xi - \eta), \quad \xi, \eta \in [0, \infty), \quad \varphi \in [0, 2\pi) \quad (5)$$

the solution of the equation (1) has the following form [1]

$$\psi_{n_1 n_2 m}^{(s)}(\xi, \eta, \varphi; \delta_1, \delta_2) = \sqrt{2\varepsilon^2} \Phi_{n_1 m_1}(\xi) \Phi_{n_2 m_2}(\eta) \frac{e^{i(m+s)\varphi}}{\sqrt{2\pi}}, \quad (6)$$

where

$$\Phi_{n_i m_i}(x) = \frac{1}{\Gamma(m_i + 1)} \sqrt{\frac{\Gamma(n_i + m_i + 1)}{(n_i)!}} e^{-\frac{\varepsilon x}{2}} (\varepsilon x)^{\frac{m_i}{2}} F(-n_i; m_i + 1; \varepsilon x).$$

The parabolic quantum numbers n_1 and n_2 are connected with the principal quantum number n as follows:

$$n = n_1 + n_2 + \frac{|m - s| + |m + s|}{2} + 1.$$

It is mentioned [1] that the parabolic basis (6) of the generalized MIC-Kepler system is the eigenfunction of commuting operators \hat{J}_z and

$$\hat{X} = \hat{I}_z + \frac{\mu}{\hbar^2} \left[\lambda_1 \frac{r - z}{r(r + z)} - \lambda_2 \frac{r + z}{r(r - z)} \right]$$

where \hat{I}_z is the z component of the analog of the Runge-Lenz vector

$$\hat{\mathbf{I}} = \frac{1}{2\sqrt{\mu}} \left[\hat{\mathbf{J}} \times \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) + \left(\hat{\mathbf{p}} + \frac{e}{c} \mathbf{A} \right) \times \hat{\mathbf{J}} \right] + \frac{e^2}{\hbar\sqrt{\mu}} \frac{\mathbf{r}}{r}$$

and

$$\hat{X} \psi_{n_1 n_2 m}^{(s)}(\xi, \eta, \varphi; \delta_1, \delta_2) = \frac{\hbar\varepsilon}{\sqrt{\mu}} \left(n_1 - n_2 + m_- + \frac{\delta_1 - \delta_2}{2} \right) \psi_{n_1 n_2 m}^{(s)}(\xi, \eta, \varphi; \delta_1, \delta_2).$$

Finally, it is mentioned also that the interbasis expansion of the parabolic basis over the spherical one has the form [1]

$$\psi_{n_1 n_2 m}^{(s)}(\xi, \eta, \varphi; \delta_1, \delta_2) = \sum_{j=m_+}^{n-1} W_{n_1 n_2 m s}^j(\delta_1, \delta_2) \psi_{n j m}^{(s)}(r, \theta, \varphi; \delta_1, \delta_2), \quad (7)$$

where

$$W_{n_1 n_2 m s}^j(\delta_1, \delta_1) = (-1)^{n_1} C_{\frac{n+m_1+\delta_2-1}{2}, \frac{m_2+n_2-n_1}{2}, \frac{n-m_1+\delta_1-1}{2}, \frac{m_1+n_1-n_2}{2}}^{j+\frac{\delta_1+\delta_2}{2}, \frac{m_1+m_2}{2}}. \quad (8)$$

Equation (8) proves that the coefficients for the expansion of the parabolic basis in terms of the spherical basis are nothing but the analytical continuation, for real values of their arguments, of the $SU(2)$ Clebsch-Gordan coefficients.

Let us demonstrate that if in equation (1) we make the changes

$$\psi^{(s)}(\mathbf{r}) \rightarrow \psi(\mathbf{r}, \gamma) = \psi^{(s)}(\mathbf{r}) \frac{e^{is(\gamma-\varphi)}}{\sqrt{4\pi}}, \quad s \rightarrow -i \frac{\partial}{\partial \gamma}, \quad \text{where } \gamma \in [0, 4\pi), \quad (9)$$

it will transform into the Schrödinger equation for a four-dimensional double singular oscillator.

Equation (1) in the spherical coordinates is of the form

$$\begin{aligned} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi^{(s)}}{\partial r} \right) + \frac{1}{r^2} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi^{(s)}}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi^{(s)}}{\partial \varphi^2} \right] - \frac{2is}{r^2(1 - \cos \theta)} \frac{\partial \psi^{(s)}}{\partial \varphi} - \\ - \frac{2s^2}{r^2(1 - \cos \theta)} \psi + \frac{2\mu}{\hbar^2} \left[E + \frac{e^2}{r} - \frac{\lambda_1}{r^2(1 + \cos \theta)} - \frac{\lambda_2}{r^2(1 - \cos \theta)} \right] \psi^{(s)} = 0. \end{aligned} \quad (10)$$

From (9) and (10) we have

$$\left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) - \frac{\hat{\mathbf{L}}^2}{r^2} \right] \psi + \frac{2\mu}{\hbar^2} \left[E + \frac{e^2}{r} - \frac{\lambda_1}{r^2(1 + \cos \theta)} - \frac{\lambda_2}{r^2(1 - \cos \theta)} \right] \psi = 0, \quad (11)$$

where

$$\hat{\mathbf{L}}^2 = - \left[\frac{1}{\sin \beta} \frac{\partial}{\partial \beta} \left(\sin \beta \frac{\partial}{\partial \beta} \right) + \frac{1}{\sin^2 \beta} \left(\frac{\partial^2}{\partial \alpha^2} - 2 \cos \beta \frac{\partial^2}{\partial \alpha \partial \gamma} + \frac{\partial^2}{\partial \gamma^2} \right) \right].$$

Here we change the notation: $\beta = \theta$ and $\alpha = \varphi$. If we now pass from the coordinates r, α, β, γ to the coordinates

$$u_0 + iu_1 = u \cos \frac{\beta}{2} e^{i\frac{\alpha+\gamma}{2}}, \quad u_2 + iu_3 = u \sin \frac{\beta}{2} e^{i\frac{\alpha-\gamma}{2}} \quad (12)$$

with $u^2 = r$, and take into account that

$$\frac{\partial^2}{\partial u_\mu^2} = \frac{1}{u^3} \frac{\partial}{\partial u} \left(u^3 \frac{\partial}{\partial u} \right) - \frac{4}{u^2} \hat{\mathbf{L}}^2, \quad \mu = 0, 1, 2, 3$$

and introduce the notations

$$\epsilon = 4e^2, \quad E = -\frac{\mu_0 \omega^2}{8}, \quad c_i = 2\lambda_i \quad (i = 1, 2)$$

equation (11) will turn into the Schrödinger equation for a four-dimensional double singular oscillator

$$\left[\frac{\partial^2}{\partial u_\mu^2} + \frac{2\mu}{\hbar^2} \left(\epsilon - \frac{\mu\omega^2 u^2}{2} - \frac{c_1}{u_0^2 + u_1^2} - \frac{c_2}{u_2^2 + u_3^2} \right) \right] \psi(\mathbf{u}) = 0, \quad (13)$$

whose energy spectrum is given by the formula

$$\epsilon = \hbar\omega (N + \delta_1 + \delta_2 + 2).$$

Using formulae (2) and (12) and considering that $r = u^2$, $\theta = \beta$, $\varphi = \alpha$, one can easily show that

$$\begin{aligned} x &= 2(u_0 u_2 - u_1 u_3), \\ y &= 2(u_0 u_3 + u_1 u_2), \\ z &= u_0^2 + u_1^2 - u_2^2 - u_3^2 \\ \gamma &= \frac{i}{2} \ln \frac{(u_0 - i u_1)(u_2 + i u_3)}{(u_0 + i u_1)(u_2 - i u_3)}. \end{aligned}$$

The first three lines are the transformation $\mathbb{R}^4 \rightarrow \mathbb{R}^3$ suggested by Kustaanheimo and Stiefel for the regularization of the equations of celestial mechanics [9]. Later, this transformation found other applications, as well [10, 11]. This transformation supplemented with the coordinate γ (generalized Kustaanheimo-Stiefel transformation) was used for the "synthesis" of the charge-dyon system from the four-dimensional isotropic oscillator [12].

Introducing the double polar coordinates

$$u_0 + i u_1 = \rho_1 e^{i\varphi_1}, \quad u_2 + i u_3 = \rho_2 e^{i\varphi_2}, \quad \text{where } \rho_1, \rho_2 \in [0, \infty), \quad \varphi_1, \varphi_2 \in [0, 2\pi). \quad (14)$$

From the formulae (2), (5), (12) and (14) we get the relations

$$\xi = 2\rho_1^2, \quad \eta = \rho_2^2, \quad \varphi = \varphi_1 + \varphi_2, \quad \gamma = \varphi_1 - \varphi_2$$

which lead to the formulae

$$\psi_{NLMM'}(u, \alpha, \beta, \gamma) = 4 \left(n + \frac{\delta_1 + \delta_2}{2} \right) \sqrt{a} \delta_{n, \frac{N}{2} + 1} \delta_{jL} \delta_{mM} \delta_{sM'} \psi_{njms}(r, \theta, \varphi, \gamma),$$

$$\begin{aligned} \psi_{N_1 N_2 M_1 M_2}(\rho_1, \rho_2, \varphi_1, \varphi_2) &= 4 \left(n + \frac{\delta_1 + \delta_2}{2} \right) \sqrt{a} \times \\ &\times \delta_{n_1, N_1} \delta_{n_2, N_2} \delta_{m, \frac{M_1 + M_2}{2}} \delta_{s, \frac{M_1 - M_2}{2}} \psi_{njms}(\rho_1, \rho_2, \varphi_1, \varphi_2) \end{aligned}$$

generalizing the earlier results [13].

Now we are able to write the expansion (7)

$$\psi_{N_1 N_2 M_1 M_2}(\rho_1, \rho_2, \varphi_1, \varphi_2) = \sum_{L=L_{min}}^{N/2} W_{N_1 N_2 M_1 M_2}^{NLMM'} \psi_{NLMM'}(u, \alpha, \beta, \gamma), \quad (15)$$

$$W_{N_1 N_2 M_1 M_2}^{NLMM'} = e^{i\pi\Phi} C_{a_0, \alpha_0; b_0, \beta_0}^{c_0, \gamma_0},$$

where

$$\begin{aligned} a_0 &= \frac{N_1 + N_2 + |M - M'| + \delta_2}{2}, & \alpha_0 &= \frac{N_2 - N_1 + |M - M'| + \delta_2}{2}, \\ b_0 &= \frac{N_1 + N_2 + |M + M'| + \delta_1}{2}, & \beta_0 &= \frac{N_1 - N_2 + |M + M'| + \delta_1}{2}, \\ c_0 &= L + \frac{\delta_1 + \delta_2}{2}, & \gamma_0 &= \frac{|M + M'| + |M - M'| + \delta_1 + \delta_2}{2}. \end{aligned}$$

The lower limit of summation in (15) and quantity Φ are given by the expressions

$$L_{min} = \frac{1}{2} (|M + M'| - |M - M'|), \quad \Phi = N_1 + \frac{1}{2} (M - M' + |M - M'|).$$

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